PUBLISHED BY INSTITUTE OF PHYSICS PUBLISHING FOR SISSA

RECEIVED: *April 21, 2005*
ACCEPTED: *June 16, 2005*
PUBLISHED: *July 20, 2005*

$c < 1$ string from two dimensional black holes

Tadashi Takayanagi

Jefferson Physical Laboratory, Harvard University
Cambridge, MA 02138, U.S.A.
E-mail: takayana@bose.harvard.edu

ABSTRACT: We study a topological string description of the $c < 1$ non-critical string whose matter part is defined by the time-like linear dilaton CFT. We show that the topologically twisted $N = 2$ $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model (or supersymmetric 2D black hole) is equivalent to the $c < 1$ non-critical string compactified at a specific radius by comparing their physical spectra and correlation functions. We examine another equivalent description in the topological Landau-Ginzburg model and check that it reproduces the same scattering amplitudes. We also discuss its matrix model dual description.

KEYWORDS: Topological Strings, Conformal Field Models in String Theory, 2D Gravity, Matrix Models.

 http://jhep.sissa.it/archive/papers/jhep072005050/jhep072005050.pdf

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1. Introduction

The two dimensional (2d) string theory has been a very useful laboratory of quantum gravity. This is because it can be exactly solvable by the dual $c = 1$ matrix model description [1, 2] including all loop corrections. Recently, remarkable progresses have been made such as the holographic interpretation of the dual matrix model [3 − 6], the non-perturbatively stable construction of the type 0 matrix model [7 − 9], the applications to time-dependent backgrounds[1] [11 − 14] and so on. Clearly, these ideas are very helpful when we would like to understand the dynamical properties of the 2d string theory as a simplest example of quantum gravity.

One of the most important unsolved problems in 2d string theory is a clear understanding of the 2d black hole [15, 16]. This background has been known to be described by the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset CFT [15] and its matrix model dual was proposed [17] in the euclidean case. Nevertheless, many of its physical properties of the black holes such as its entropy etc. have been poorly understood even at present[2]. One interesting way to make an important

[1] Early relevant works can be found in [10].

[2] For recent progresses on this subject refer to e.g. [18 − 27].

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progress in such a difficult problem in string theory is to direct our attention to its topological properties. For example, when we are interested in the critical superstring compactified on a Calabi-Yau space, we can extract important holomorphic quantities by considering the topological string [28, 29] on that space. In the same way, we can expect that the topologically twisted $N = 2$ $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model (Kazama-Suzuki model [30]) captures certain significant properties of 2d black holes in superstring [17, 31, 18]. In this case, we have an advantage that the topological twisting does not reduce the degree of freedom so much. This is because the 2d string theory itself does not include infinitely many massive fields as opposed to the ordinary ten dimensional superstring. Indeed, it has been known that such kinds of lower dimensional string theories (i.e. non-critical strings) often possess equivalent descriptions in terms of topological strings [32].

Such an example, which is the most relevant to us, will be the well-known equivalence [33, 34] between the twisted $N = 2$ $\mathrm{SL}(2,\mathbb{R})_3/\mathrm{U}(1)$ coset model at the level $k = 3$, and the $c = 1$ string (or equally the 2d string) compactified at the self-dual radius. This twisted theory can be geometrically interpreted as the topological string on the conifold [35, 36]. It is also known to be equivalent[3] to the topological Landau-Ginzburg (LG) model defined by the superpotential $W = X^{-1}$ [39, 40, 36, 41]. However, in this example its relation to the physical type 0 string on the 2d black hole is not so clear since its critical condition requires the different value $k = 3/2$ of the level.

Motivated by this, we would like to discuss the topologically twisted $N = 2$ $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ model at general values of the level k in the present paper. We will argue that it is equivalent to a specific compactified $c < 1$ string whose matter part is defined by the time-like linear dilaton CFT [14] instead of minimal models. We will often call this theory a non-minimal $c < 1$ string below just for simplicity. The results in this paper are conveniently summarized in the final section. If we look at this equivalence reversely, it also reveals the essential topological properties of the $c < 1$ string theory.

In this $c < 1$ string, the matter CFT is described by a time-like boson X_0 whose linear dilaton gradient is $q = \sqrt{2}(1/b - b)$. Thus the central charge of this matter CFT is $c_X = 1 - 6q^2$. The Liouville sector is defined in a standard way by the space-like scalar ϕ whose linear dilaton gradient is $Q = \sqrt{2}(b + 1/b)$ (central charge $c_\phi = 1 + 6Q^2$). We also assume the usual Liouville term $\mu \int dz^2 e^{\sqrt{2}b\phi}$. Totally this model defines a critical bosonic string because $c_X + c_\phi = 26$. After we perform a Lorentz boost so that the coupling constant g_s does not depend on time, it represents a simple time-dependent background in 2d string theory with a non-standard Liouville potential $\mu \int dz^2 e^{[(b^2-1)X^0+(1+b^2)\phi]/\sqrt{2}}$ [14]. Indeed the matrix model dual of this background is given by the following time-dependent fermi surface [14] of $c = 1$ matrix model in the phase space (x, p)

$$\left(\frac{-p-x}{2}\right)^{b^2}\left(\frac{p-x}{2}\right) = \mu e^{(b^2-1)t}\,, \tag{1.1}$$

where t is the time in the matrix model. The special property of this model compared with other time-dependent ones is that it is solvable even in the world-sheet theory.

[3]This equivalence is essentially the same as the mirror symmetry which relates the $N = 2$ Liouville theory with the $N = 2$ $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model [37, 17, 31, 38].

The deformation parameter b in the $c < 1$ string is identified with the one in the twisted model via the relation $n = b^{-2}$ (or equally $n = b^2$) under the equivalence. We also see that these have another equivalent description by the topological LG models defined by the potential $W = X^{-n}$. In a particular limit $b = 1$ we reproduce the known equivalence between $c = 1$ string and the topological models [33, 34, 42, 39, 40]. The possibility of the equivalence between the twisted $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model and a certain (p,q) non-critical string was already mentioned in [42, 43, 36] when n is a rational number $n = p/q$. In the present paper, we explicitly identify the relevant string theory with the compactified non-minimal $c < 1$ string defined just before. This string theory can also be defined by the dual matrix model description [14]. In addition, we also find that most of our results do not depend on whether n is rational or irrational except the periodicity in the euclidean time direction.

Another aim of this paper is to check this kind of equivalence between the topological strings and the non-critical strings including non-trivial interactions in addition to the comparison of the physical spectrum. One way to compute the interactions is to examine the correlators by employing the mathematical structures of the topological gauged WZW model [33]. It is also possible to do this by using the equivalent description of the topological LG models [39, 40]. Even though these analyses reproduce the essential part of the scattering amplitudes in the non-critical string, the derivation of the non-local transformation (or almost equally so called leg factor [2]) for each vertex operators is not straightforward. However, now we can also directly compute the correlators in terms of those in the untwisted theory owing to the recent progresses of the understanding of $\mathrm{SL}(2,\mathbb{R})$ WZW model (refer to [44, 45] and references therein). This has not been done even for the familiar $c = 1$ case [34]. In the present paper we will explicitly compute the three point functions in the twisted coset model and check that they agree with those in the $c < 1$ string including the $c = 1$ case as a particular limit.

The organization of this paper is as follows. In section 2, we examine the twisted $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model. We find its physical states in the free field representation and show that it is the same as those in the $c < 1$ string. In section 3, we directly compute the three point function in the twisted coset model and check that they agree with those in the $c < 1$ string. In section 4, we discuss the topological LG model, which is expected to be equivalent to the twisted $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model. Indeed we check the model reproduces the correct scattering amplitudes of tachyons in the $c < 1$ string. In section 5 we summarize the conclusions and discuss future problems.

2. Twisting 2D black holes

We would like to investigate the topological (A-model) twist of the $N = 2$ coset model $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}$. This SCFT describes the supersymmetric version of 2d black hole [15–17]. What we would like to show is that the spectrum of physical states agrees with that of the non-minimal $c < 1$ string [14] whose matter part is defined by the time-like linear dilaton theory. Since the argument here is a natural extension of those in the $c = 1$ string case [34], the discussions below will be a bit brief and we will share almost the same notations. Never-

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theless, we will also add several clarifications in the light of the modern understandings of the SL$(2,\mathbb{R})$ WZW model (see [36] and references therein). We use the normalization of OPEs in the $\alpha'=2$ unit in the most part of this paper.

2.1 Description of coset model

The $N=2$ $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)}$ coset model[4] at level $n(>0)$ is equivalent to the product of the bosonic $\mathrm{SL}(2,\mathbb{R})_{n+2}$ WZW model (at level $k=n+2$) and a Dirac free fermion $(\psi,\bar{\psi})$. In this section we do not have to assume that n is an integer. Whether it is an integer or not becomes relevant when we consider its interpretation as a non-critical string theory. We will discuss this issue in section 5.2.

We employ the Wakimoto free field representation [46] for the bosonic $\mathrm{SL}(2,\mathbb{R})_{n+2}$ WZW model via the bosonization[5]

$$J^-=\beta\,, \qquad J^+=\beta\gamma^2-\sqrt{2n}\gamma\partial\phi+(n+2)\partial\gamma\,, \qquad J^3=\beta\gamma-\sqrt{\frac{n}{2}}\partial\phi\,. \tag{2.1}$$

The OPEs are defined by $\phi(z)\phi(0)\sim-\log z$, $\beta(z)\gamma(0)\sim 1/z$. The operators (2.1) satisfy the $\mathrm{SL}(2,\mathbb{R})_{n+2}$ current algebra

$$J^3(z)J^3(0)\sim-\frac{(n+2)/2}{z^2}\,, \quad J^3(z)J^{\pm}(0)\sim\pm\frac{J^{\pm}(z)}{z^2}\,, \quad J^+(z)J^-(0)\sim\frac{n+2}{z^2}-\frac{2J^3(0)}{z}\,. \tag{2.2}$$

The stress energy tensor is given by

$$T=\beta\partial\gamma-\frac{1}{2}(\partial\phi)^2+\frac{1}{\sqrt{2n}}\partial^2\phi\,. \tag{2.3}$$

The primary fields $\Phi_{j,m}$ in the $\mathrm{SL}(2,\mathbb{R})$ current algebra are expressed as (m is the eigenvalue of J_0^3)

$$\Phi_{j,m}=\gamma^{j+m}e^{-\sqrt{\frac{2}{n}}j\phi}\,. \tag{2.4}$$

Then we can construct the lowest weight discrete representation $D_j^+ : m=-j,-j+1,-j+2,\dots$ starting from the lowest weight state (LWS) $\Phi_{j,-j}=e^{-\sqrt{\frac{2}{n}}j\phi}$. We can also find its conjugate representation by identifying its LWS with

$$\Phi'_{j,-j}=(\beta)^{n+2j+1}e^{\sqrt{\frac{2}{n}}(j+n+1)\phi}\,. \tag{2.5}$$

On the other hand, the highest weight discrete representation is denoted by $D_j^- : m=j,j-1,j-2,\cdots$. Its highest weight state (HWS) has the spin $m=j$. Since the background charge[6] of ϕ is $Q(\phi)=\sqrt{\frac{2}{n}}$ as can be seen from (2.3), we can check that the conformal dimension of the operators $\Phi_{j,m}$ and $\Phi'_{j,m}$ is $\Delta=-\frac{j(j+1)}{n}$.

---

[4]We can also write the supersymmetric model as $\frac{\mathrm{SL}(2,\mathbb{R})_{n+2}\times\mathrm{U}(1)}{\mathrm{U}(1)}$. The U(1) part in the numerator corresponds to the fermions and we write this by ψ and $\bar{\psi}$.

[5]In the most part of this paper we make explicit only the left-moving (or chiral) sector. The right-moving (anti-chiral) sector can be constructed in the same way. The world-sheet coordinate is denoted by $(z,\bar{z})$.

[6]We define the background charge for a field ϕ by $T=-\frac{1}{2}(\partial\phi)^2+\frac{Q}{2}\partial^2\phi$. The string coupling constant behaves like $g_s=e^{\frac{Q}{2}\phi}$. The conformal dimension of the primary field $e^{\alpha\phi}$ is given by $\Delta(e^{\alpha\phi})=\frac{1}{2}\alpha(Q-\alpha)$.

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In order to define the coset, we can gauge the following U(1) current (see e.g. [47, 36])

$$J_g = J^3 - \bar{\psi}\psi - i\sqrt{\frac{n}{2}}\partial X = \hat{J}^3 - i\sqrt{\frac{n}{2}}\partial X\,, \tag{2.6}$$

where $\hat{J}^3$ is the third current of the super $\mathrm{SL}(2,\mathbb{R})_n$ $N=1$ WZW model. Note also the OPEs $X(z)X(0) \sim -\log(z)$ and $\psi(z)\bar{\psi}(0) \sim \frac{1}{z}$. To perform the U(1) quotient we can add the $c=-2$ ghosts[7] (ξ,η) and define the BRST charge $Q_B = \int dz \xi(z) J_g(z)$. The U(1) current of the $N=2$ SCFT is

$$J_R = \frac{n+2}{n}\bar{\psi}\psi - \frac{2}{n}J^3 \simeq \bar{\psi}\psi - i\sqrt{\frac{2}{n}}\partial X\,. \tag{2.7}$$

Notice that $J_q(z)J_R(0) \sim 0$. Then we can equivalently use the following U(1) current of the $N=2$ SCFT (see also [36])

$$J'_R = J_R - 2J_g = 3\bar{\psi}\psi - 2J^3 + i\sqrt{\frac{2}{n}}(n-1)\partial X\,. \tag{2.8}$$

Geometrically the quotient $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ looks like a cigar (or 2d black hole) with the asymptotic radius $R=\sqrt{2n}$.

2.2 Topological twist

Now let us take the topological (A-model[8]) twist [29] of the $N=2$ coset via the standard rule $T \to T + \frac{1}{2}\partial J$ [48]. The background charges become $Q'(\phi) = \sqrt{\frac{2}{n}} + \sqrt{2n}$ and $Q'(X) = i(\sqrt{2n} - \sqrt{\frac{2}{n}})$. The central charges become $c = 1 + 6\frac{(1+n)^2}{n}$ and $c = 1 - \frac{6(n-1)^2}{n}$, respectively. In summary we started with the fields ($\Delta(A)$ denotes the conformal dimension of the operator A)

$$\begin{aligned}
X:&\quad c=1\,,\quad Q(X)=0\\
\phi:&\quad c=1+\frac{6}{n}\,,\quad Q(\phi)=\sqrt{\frac{2}{n}}\\
(\psi,\bar{\psi}):&\quad c=1\,,\quad \Delta(\psi)=\Delta(\bar{\psi})=\frac{1}{2}\\
(\beta,\gamma):&\quad c=2\,,\quad \Delta(\beta)=1\,,\quad \Delta(\gamma)=0(\eta,\xi):\quad c=-2\,,\quad \Delta(\eta)=1\,,\ \Delta(\xi)=0\,,
\end{aligned} \tag{2.9}$$

and after the twist we get

$$X:\quad c = 1-\frac{6(n-1)^2}{n}\,,\quad Q(X) = i\left(\sqrt{2n}-\sqrt{\frac{2}{n}}\right)$$

[7]Their conformal dimensions are $\Delta(\xi)=0$ and $\Delta(\eta)=1$.

[8]Here we define the topological string theory by the A-model twist i.e. $T(z) \to T(z) + \frac{1}{2}\partial J(z)$ and $T(\bar{z}) \to T(\bar{z}) - \frac{1}{2}\bar{\partial}J(\bar{z})$ [29, 48]. The anti-chiral (or right-moving) counterpart of the R-current (2.8) is given by $J'_R(\bar{z}) = 3\bar{\psi}\psi + 2J^3 + i\sqrt{\frac{2}{n}}(n-1)\bar{\partial}X$ in our convention; the right-moving gauging current is $J_g(\bar{z}) = J^3(\bar{z}) + 3\psi\bar{\psi} + i\sqrt{\frac{2}{n}}\bar{\partial}X$. Thus the twist acts on the $\mathrm{SL}(2,\mathbb{R})$ part symmetrically, while it does on the U(1) boson X asymmetrically. Notice that we do not want to perform the B-model twist because in that case the background charge for ϕ in the left-moving section takes the sign opposite to the one in the right-moving sector.

$$
\begin{aligned}
\phi: &\quad c = 1 + \frac{6(1+n)^2}{n}\,, \quad Q(\phi) = \sqrt{\frac{2}{n}} + \sqrt{2n} \\
(\psi, \bar\psi): &\quad c = -26\,, \quad \Delta(\psi) = 2\,, \quad \Delta(\bar\psi) = -1 \\
(\beta, \gamma): &\quad c = 2\,, \quad \Delta(\beta) = 0\,,\ \Delta(\gamma) = 1 (\eta, \xi): \quad c = -2\,, \quad \Delta(\eta) = 1\,,\ \Delta(\xi) = 0\,.
\end{aligned} \tag{2.10}
$$

Notice that the total central charge after the twist is zero as expected.

Then we can interpret the twisted system as a critical bosonic string. The boson X is a free boson with the linear dilaton and ϕ is the Liouville field on the world-sheet. In the 2d spacetime viewpoint, X and ϕ are the (euclidean) time and space coordinate. The fermions $(\psi, \bar\psi)$ correspond to the (b, c) ghosts. The fields (β, γ) and (η, ξ) are almost canceled with each other. The original screening term in the $\mathrm{SL}(2, \mathbb{R})$ WZW model

$$
\int dz^2 \beta(z) \tilde\beta(\bar z) e^{\sqrt{\frac{2}{n}}\phi}\,, \tag{2.11}
$$

can be regarded as the Liouville potential term in the $c < 1$ string

$$
\mu \int dz^2 e^{\sqrt{\frac{2}{n}}\phi}\,, \tag{2.12}
$$

since β becomes conformal dimension zero after the twist and can be treated as a constant. On the other hand, there is no screening term for the X field. Thus this string theory is equivalent to the non-minimal $c < 1$ string [14] (also this is reviewed in the introduction) at $b = \frac{1}{\sqrt{n}}$ (or equally $b = \sqrt{n}$). Notice also the euclidean 'time' X is compactified and its radius[9] is again given by $R = \sqrt{2n}$.

2.3 Chiral primaries

Here we would like to find the chiral primaries in the coset $N = 2$ SCFT because they are obvious candidates for physical states in the topologically twisted theory. Before we go on, we define several free bosons to make notations simple.[10] First let us bosonize the fermion as

$$
\psi(z) = e^{iH(z)}\,, \qquad \bar\psi(z) = e^{-iH(z)}\,, \qquad \psi(z)\bar\psi(z) = i\partial H(z)\,. \tag{2.13}
$$

We also define bosons X_3 and X_R from $J^3(z)$ and $J_R(z)$ as follows

$$
J^3(z) = -\sqrt{\frac{n+2}{2}}\partial X_3(z)\,, \qquad J_R(z) = -i\sqrt{\frac{n+2}{n}}\partial X_R(z)\,. \tag{2.14}
$$

We can also rewrite (2.7) as follows

$$
X_R = \sqrt{\frac{n+2}{n}} H + i\sqrt{\frac{2}{n}} X_3\,. \tag{2.15}
$$

These bosons are normalized such that their OPEs are given by $H(z)H(0) \sim -\log(z)$, $X_3(z)X_3(0) \sim -\log(z)$ and $X_R(z)X_R(0) \sim -\log(z)$.

[9]Below we also consider the $\mathbb{Z}_n$ orbifold of the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. In that case we have the different radius $R = \sqrt{2/n}$.

[10]We will closely follow the notations in [18]. However, the definition of J_R has opposite sign to [18].

The primary fields $\Phi_{j,m}$ in the bosonic SL$(2,\mathbb{R})$ WZW can be expressed as

$$\Phi_{j,m} = V_{jm} e^{\sqrt{\frac{2}{n+2}} m X_3} , \qquad (2.16)$$

where V_{jm} is the primary of SL$(2,\mathbb{R})/$ U(1) coset. The conformal dimensions are $\Delta(\Phi_{j,m}) = -\frac{j(j+1)}{n}$. $\Delta(V_{jm}) = -\frac{j(j+1)}{n} + \frac{m^2}{n+2}$. Notice that the V_{jm} part has no J_0^3 charge.

Let us return to the formulation with ghosts (ξ, η). Because of the BRST invariance[11] $Q_B = \int dz \xi(z) J_g(z) \sim 0$ on the physical states (see (2.6)), the primary operators

$$e^{isH} V_{jm} e^{\sqrt{\frac{2}{n+2}} m X_3} , \qquad (2.17)$$

in $N = 1$ SL$(2,\mathbb{R})$ model, which has the J_g charge $q_g = s + m$ and the dimension $\Delta = -\frac{j(j+1)}{n} + \frac{s^2}{2}$, are now dressed by X as follows

$$e^{isH} V_{jm} e^{\sqrt{\frac{2}{n+2}} m X_3} \cdot e^{i\sqrt{\frac{2}{n}}(s+m)X} . \qquad (2.18)$$

The role of X dressing is that it annihilates with the unwanted U(1) part in N=1 SL$(2,\mathbb{R})$ operator (2.17) . We can find their conformal dimensions

$$\Delta = \left(-\frac{j(j+1)}{n} + \frac{s^2}{2} \right) + \frac{(s+m)^2}{n} . \qquad (2.19)$$

Chiral primary states in the NS-sector of the $N = 2$ coset SCFT satisfy

$$G^+_{-1/2} |NSc\rangle = 0 . \qquad (2.20)$$

The $N = 2$ superconformal generators in the coset model [30] are given by

$$G^+(z) = \sqrt{\frac{2}{n}} \bar{\psi}(z) J^+(z) , \qquad G^-(z) = \sqrt{\frac{2}{n}} \psi(z) J^-(z) . \qquad (2.21)$$

We denote the operator which corresponds to the state $|NSc\rangle$ by O_{NSc}. Requiring the condition (2.20) or $G^+(z) O_{NSc}(0) \sim \frac{0}{z}$, we can find the following chiral primaries (i.e. $s = 0$ and $j = m$)

$$O_{NSc} = V_{jj} e^{\sqrt{\frac{2}{n+2}} j X_3} \cdot e^{i\sqrt{\frac{2}{n}} j X} . \qquad (2.22)$$

This has the conformal weight $\Delta = -j/n$ and R-charge $q_R = -2j/n$ (in the same way we can construct anti-chiral state defined by $s = 0$ and $j = -m$ with $\Delta = -q_R/2 = -j/n$).

We would like to perform the spectral flow[12] so that we get R-sector states $|Rc\rangle$ [49] that satisfy

$$G^+_0 |Rc\rangle = G^-_0 |Rc\rangle = 0 . \qquad (2.23)$$

Notice that the first condition shows that this is the physical state in the topological twisted model which we are interested in.

[11] Note that this BRST operator is completely different from the one which defines the topological twisted theory.

[12] Note the formula $\Delta' = \Delta + \theta q_R + c\theta^2/6$, $q'_R = q_R + c\theta/3$ for the spectral flow by the angle θ [49]. This corresponds to the multiplication of the operator $e^{-i\theta\sqrt{\frac{n+2}{n}}X}$. In particular we are interested in the spectral flow from NS to R-sector is $\theta = -1/2$. Also distinguish this spectral flow from the one we perform in the next subsection.

The spectral flow corresponds to the shift of X_R momentum by $e^{\frac{i}{2}\sqrt{\frac{n+2}{n}}X_R} \sim e^{\frac{i}{2}H+\frac{i}{\sqrt{2n}}X}$. Thus we find the corresponding operators in R-sector

$$V_{j,j} e^{\sqrt{\frac{2}{n+2}} j X_3} \cdot e^{\frac{i}{2}H} e^{i\sqrt{\frac{2}{n}}(j+1/2)X} . \tag{2.24}$$

These have the property $\Delta = \frac{n+2}{8n}$ and $q_R = -\frac{2j}{n} - \frac{n+2}{2n}$. Note that the second condition in (2.23) is too restrictive to find the physical state and there are indeed other states as we will explain later. It is also a useful fact that the non-triviality of cohomology with respect to G_0^+ requires the condition $\Delta = \hat{c}/8 = \frac{n+2}{8n}$.

2.4 Physical states

To find the physical states in the topologically twisted theory, we can start with the Ramond states which satisfy $G_0^+|Rc\rangle = 0$ in the untwisted model and perform the spectral flow [49, 34] into the topologically twisted one (i.e. the states in NS-sector).

The physical states corresponding to the R-states (2.24) can be found as follows. Because of the twisted background charges of various bosons, the spectral flow based on the R-current (2.8) is defined by

$$|0\rangle_r = e^{-\frac{i}{2}\sqrt{\frac{n+2}{n}}X_R}|0\rangle_t \sim e^{-\frac{3}{2}iH} e^{i(\sqrt{\frac{n}{2}}-\frac{1}{\sqrt{2n}})X} e^{\sqrt{\frac{n+2}{2}}X_3}|0\rangle_t \,, \tag{2.25}$$

where $|0\rangle_r$ is the Ramond sector vacuum before the twisting, and $|0\rangle_t$ is the vacuum of the topological theory. From this argument it is obvious that the total expressions of the physical states will include the factors

$$e^{-iH} \cdot e^{\sqrt{\frac{2}{n}}[i(j+\frac{n}{2})X]}. \tag{2.26}$$

To find the result for the bosonic $\mathrm{SL}(2,\mathbb{R})$ WZW part, notice that the procedure (2.25) is just the same as the $w=1$ spectral flow of $\mathrm{SL}(2,\mathbb{R})$ WZW model [45] (see the appendix for a brief review of the spectral flow[13]). Since we started with the HWSs (highest weight states) $\Phi_{j,j}$ in D_j^-, after the spectral flow it becomes the LWSs (lowest weight states) in $D_j^{-,w=1}$. Because $D_j^{-,w=1}$ are equivalent to $D^+_{-j-\frac{n+2}{2}}$ [45], the states after the spectral flow can be written as

$$\Phi_{-j-\frac{n+2}{2},\, j+\frac{n+2}{2}} = e^{\sqrt{\frac{2}{n}}(j+\frac{n+2}{2})\phi} \,, \tag{2.27}$$

in the Wakimoto representation (2.4).

These arguments lead to the following physical states in the topological model

$$V_j = c\; e^{\sqrt{\frac{2}{n}}[i(j+\frac{n}{2})X + (j+\frac{n+2}{2})\phi]}, \tag{2.28}$$

where we have used the fact that $\bar{\psi} = e^{-iH}$ can be identified with the ghost c after the twist (2.10). Indeed these operators have the vanishing conformal dimension. We can also rewrite them in the following way so that they look the same as the tachyon vertex

[13]Refer also to [50] for the spectral flow in the free field representation.

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operators in the $c < 1$ string

$$V_j = c\, e^{\frac{Q(X)}{2}X + \frac{Q(\phi)}{2}\phi} \cdot e^{\sqrt{\frac{2}{n}}(j+\frac{1}{2})(iX+\phi)}\,. \tag{2.29}$$

The quantum number j in our model[14] takes all half integers ($2j \in \mathbb{Z}$) in the left-right symmetric sector (momentum modes), while in the asymmetric sector (winding modes) we have the different condition $2j \in nZ$. This means that the radius of X is $R_A = \sqrt{2n}$. It is also useful to consider a $\mathbb{Z}_n$ orbifolded model, where its $\mathbb{Z}_n$ action acts as the shift $X \to X + 2\pi\sqrt{2/n}$. In this case the radius is $R_A = \sqrt{2/n}$ in the $\alpha' = 2$ unit.

Then it is clear that the operator (2.29) can be regarded as the tachyon vertex with the momentum $p_X = \sqrt{\frac{2}{n}}(j + 1/2)$ in the non-minimal $c < 1$ string compactified at the radius $R = \sqrt{2/n}$ after we take T-duality[15] in the X direction. If we consider the $\mathbb{Z}_n$ orbifolded model, then we get the radius $R = \sqrt{2n}$ for the $c < 1$ string.

It is possible to find other physical states by acting $\mathrm{SL}(2,\mathbb{R})$ currents almost in the same way as in the $n = 1$ case [34]. For $j \leq -\frac{n+2}{2}$, we can act $(J^+_{-1})^{-2j-1-n} \sim (\gamma_{-1})^{-2j-1-n}$ on V_j such that $\Delta = 0$ and obtain the following physical states

$$B_j = c\gamma^{-2j-n-1} e^{-i\sqrt{\frac{2}{n}}(j+\frac{n+2}{2})X} e^{\sqrt{\frac{2}{n}}(j+\frac{n+2}{2})\phi}\,. \tag{2.30}$$

We can also find similar physical states[16] by acting G^-_0

$$B'_j = \gamma^{-2j-n-2} e^{-i\sqrt{\frac{2}{n}}(j+\frac{n+2}{2})X} e^{\sqrt{\frac{2}{n}}(j+\frac{n+2}{2})\phi}\,. \tag{2.31}$$

For $j \geq 0$ we can act $(J^-_0)^{2j+1} = (\beta_0)^{2j+1}$ such that $\Delta = 0$ and find the physical states

$$\tilde{V}_j = c\beta^{2j+1} e^{-i\sqrt{\frac{2}{n}}(j-\frac{n}{2}+1)X} e^{\sqrt{\frac{2}{n}}(j+\frac{n+2}{2})\phi}\,. \tag{2.32}$$

This is equivalent to the conjugate LWS operators $\Phi'_{j-n/2,-j+n/2}$ (see (2.5)) in $D^+_{j-n/2}(= D^{-,w=1}_{-j-1})$. To interpret the operators (2.32) as tachyon vertexes in the $c < 1$ string, we can neglect the power of β. This is possible since $\Delta(\beta) = 0$ as in the arguments around (2.11) and (2.12). Then they looks like

$$\tilde{V}_j \sim c\, e^{\frac{Q_X}{2}X + \frac{Q_\phi}{2}\phi} \cdot e^{\sqrt{\frac{2}{n}}(j+\frac{1}{2})(-iX+\phi)}\,. \tag{2.33}$$

[14]In the standard discussions of the $\mathrm{SL}(2,\mathbb{R})$ WZW model or its U(1) coset, we usually assume the unitarity bound [45, 37, 31, 51, 38]. This is given by in our convention $-(n+1)/2 < j < -1/2$. However, in this paper we do not require this condition, so as to make the physical spectrum rich enough for our purpose. The similar treatment seems to be valid even before the topological twisting. One way to understand this claim intuitively is to consider the 2d black hole ($\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model) as a background in 2d string and try to find its physical spectrum (see e.g. [52]) of tachyon field. It should match with the conventional two dimensional string in the asymptotic region $\phi \to -\infty$. Thus we need to consider the same theory without the condition since there are no bound for the momentum of the tachyon field in the 2d string (see also [18]). Notice also that it is possible to neglect the unitarity bound in the large n limit (i.e. the classical limit) without any such assumptions since the condition $j < -1/2$ means the standard non-normalizability or the Seiberg bound [53].

[15]Here we took the T-duality so that we have the same linear-dilaton gradient in both left and right-moving direction. Notice that the A-twist leads to a left-right asymmetric extra linear dilaton.

[16]Notice that B'_j is the HWS $\Phi_{-j-\frac{n+2}{2},-j-\frac{n+2}{2}}$ in $D^-_{-j-\frac{n+2}{2}}(= D^{+,w=-1}_j)$.

Thus it is obvious from (2.29) and (2.33) that the tachyon operators $\tilde{V}_j$ are dual to V_{-j-1} due to the reflection at the Liouville wall; i.e. $\tilde{V}_j \sim V_{-j-1}$. This relation can also be independently seen from the $\mathrm{SL}(2,\mathbb{R})$ model side by noting that $\tilde{V}_j$ and V_j are regarded as the primary fields $\Phi_{j,j}$ and $\Phi_{j,-j-1}$ in D_j^- in the Ramond state description like (2.24). The operators $\Phi_{j,m}$ are equivalent to $\Phi_{-j-1,m}$ via the reflection relation [44, 37]

$$\Phi_{j,m} \simeq R(j,m,m)\Phi_{-j-1,m}\,. \tag{2.34}$$

The coefficient $R(j,m,m)$ will be explicitly given in the section 3.1.

In this way we have shown that the tachyon states at the ghost number one (i.e. $Y^{\pm}_{s,\pm s}$ in the conventional notation [54]) can all[17] be obtained from the physical states V_j and $\tilde{V}_j$ in the twisted $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model. It is also possible to see that the operators B_j and B'_j can be understood as specific ground ring states [55, 54] (with ghost number one and zero, respectively). In the conventional notation we can write them as $\tilde{a}O_{s,s}$ and $O_{s,s}$, where $\tilde{a} \equiv c\gamma$.

To get the complete spectrum [55] , we also need to combine the LWS in D_j^+ in addition to the HWS in D_j^- when we first start with the Ramond state description (2.24) in the untwisted theory as in [34]. Starting with $\Phi_{-j,j}$ in the Ramond state with the opposite spin $e^{-\frac{i}{2}H}$, after the spectral flow we can find

$$T_j = c\partial c\, e^{\sqrt{\frac{2}{n}}[i(j+\frac{n}{2}-1)X+(j+\frac{n}{2})\phi]}\delta(\beta)\,, \tag{2.35}$$

where $\delta(\beta)$ is defined by $e^{\tilde{\phi}}$ in terms of the bosonized field of $\beta\gamma$ system (see the appendix). The operators (2.35) have the ghost number two and are one to one correspondence to the ghost number one counterpart V_{j-1} in the two dimensional string theory. We can also find by acting $(J_0^+)^{-2j+1}$ for $j \leq 0$ on (2.35)

$$\tilde{T}_j = c\partial c\ (\beta)^{2j-2} e^{\sqrt{\frac{2}{n}}[-i(j-\frac{n}{2})X+(j+\frac{n}{2})\phi]}\,, \tag{2.36}$$

which are the ghost number two counterpart of $\tilde{V}_{j-1}$. Thus we can obtain the tachyon states[18] at ghost number two (i.e. $\tilde{a}Y^{\pm}_{s,\pm s}$) in the $c<1$ string as T_j and $\tilde{T}_j$.

We can show the relation $T_j \sim \tilde{T}_{-j+1}$ from (2.34) as before. Indeed there are non-vanishing two point functions $\langle V_j T_{-j}\rangle$, $\langle \tilde{V}_{-j-1}T_{-j}\rangle$, $\langle V_j\tilde{T}_{j+1}\rangle$ and $\langle \tilde{V}_{-j-1}\tilde{T}_{j+1}\rangle$ as is clear from the reflections due to Liouville wall in the $c<1$ string. They can also be computed by using the known expression of the two point functions in $\mathrm{SL}(2,\mathbb{R})$ WZW model as we will discuss in the section 3.2.

To find the discrete states [55] in the $c<1$ string, we can again apply the method taken in the $c=1$ string case [34]. Define the operator called K^- by boosting our model into the $c=1$ string background

$$K^- = \beta e^{-i\sqrt{2}X'}\,, \tag{2.37}$$

[17] To be correct, there is the bound $j \geq 0$ for $\tilde{V}_j$. The other states with $j<0$ (i.e. $Y^{+}_{s,s}$) can be obtained by acting G_0^- on $\tilde{T}_{j+1}$ (for the definition of $\tilde{T}_j$ see the arguments below).

[18] To be correct, we cannot find $\tilde{a}Y^{-}_{s,-s}$ because of the bound $j \leq 0$. It comes from the previous HWS tachyon state $Y^{-}_{s,s} = V_{ns-1/2}$ by acting $\tilde{a}(K^-)^{2s}$.

where X' is the euclidean time in the $c = 1$ string. It is defined by the boost

$$\begin{aligned} iX' &= \frac{i}{2}\left(\sqrt{n}+\frac{1}{\sqrt{n}}\right)X + \frac{1}{2}\left(\sqrt{n}-\frac{1}{\sqrt{n}}\right)\phi\,, \\ \phi' &= \frac{i}{2}\left(\sqrt{n}-\frac{1}{\sqrt{n}}\right)X + \frac{1}{2}\left(\sqrt{n}+\frac{1}{\sqrt{n}}\right)\phi\,. \end{aligned} \tag{2.38}$$

By acting K^- on the previous tachyon states and the ground ring states, we can find all physical states[19] in the $c < 1$ string including the discrete states and ground ring states $Y^{\pm}_{s,n}$, $\tilde{a}Y^{\pm}_{s,n}$, $O_{s,n}$ and $\tilde{a}O_{s,n}$. Notice that the action of K^- on V_j is only well-defined when $2j + 1 \in nZ$ due to the requirement of the locality of their OPE.

In this way we have checked that the physical spectrum of the twisted $\mathrm{SL}(2,\mathbb{R})_{2+n}/\mathrm{U}(1)$ model (or its $\mathbb{Z}_n$ orbifold) coincides with that of the non-minimal $c < 1$ string compactified at the radius $R = \sqrt{\frac{2}{n}}$ (or $R = \sqrt{2n}$).

3. Three point functions

Now we would like to move on to the analysis of interactions in the twisted coset theory. We will compute the two and three point functions and check that they are matching with the scattering amplitudes in the $c < 1$ string. What we have to show for the consistency is that the correlation functions agree with each other up to certain non-local factors (i.e. normalization) for each vertex operators. Furthermore, as we will see later, these momentum dependent factors are essentially the same as the leg-factors (see e.g. [2]) in the $c < 1$ string computed in [56, 14]. As a specific limit, our results in this section also provide a new evidence for the analogous equivalence in $c = 1$ string [34].

3.1 Two and three point functions in $\mathrm{SL}(2,\mathbb{R})$ WZW model

First we review the known results of two and three point functions in $\mathrm{SL}(2,\mathbb{R})_{k=n+2}$ WZW model [44, 37] since they are the essential parts of the correlation functions in the twisted coset theory. We will follow the convention in [37, 57]. Notice that in order to shift the convention in the previous section to the one in this section, one has[20] to change the sign: $\phi \to -\phi$. In this subsection we write the primaries in bosonic $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ by $V_{j,m,\bar{m}}$ (see (2.16)), where m and $\bar{m}$ are the eigenvalues of J^3_0 in the left and right-moving sector. Note that the correlators of $V_{j,m,\bar{m}}$ are essentially the same as those in the bosinic $\mathrm{SL}(2,\mathbb{R})$ WZW model.

[19]The definition of $Y^{\pm}_{s,n}$ and $O_{s,n}$ are given by in terms of $c = 1$ fields (see [34]) $Y^{\pm}_{s,n} = c(K^-)^{s-n}\exp(i\sqrt{2}X' + \sqrt{2}(1\mp s)\phi')$ and $O_{s,n} = (K^-)^{s-n}\gamma^{2s}\exp(i\sqrt{2}X' - \sqrt{2}\phi') = \beta^{-s-n}x^{s+n}y^{s-n}$, where x and y is the standard ground generators.

[20]This is because in [37, 57], the coupling constant is defined by $g_s = e^{-\frac{Q}{2}\phi}$. On the other hand, the field X has the same sign as in the previous section.

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The non-trivial two point function of primaries are given by the following formula[21]

$$\begin{aligned}&\langle V_{j,m,\bar m}V_{j,-m,-\bar m}\rangle \quad (\equiv R(j,m,\bar m)) = \\ &= n(\nu)^{2j+1}\frac{\Gamma(1-(2j+1)/n)\Gamma(-2j-1)}{\Gamma(\frac{2j+1}{n})\Gamma(2j+2)}\cdot\frac{\Gamma(j-m+1)\Gamma(1+j+\bar m)}{\Gamma(-j-m)\Gamma(\bar m-j)}\,,\end{aligned} \tag{3.1}$$

where ν is defined by $\nu=\frac{\Gamma(1+1/n)}{\pi\Gamma(1-1/n)}$. Remember that in addition we have the trivial ones $\langle V_{j,m,\bar m}V_{-j-1,-m,-\bar m}\rangle=1$. Notice also that $R(j,m,m)$ is equal to the reflection coefficient in (2.34).

The three point functions are given[22] by

$$\begin{aligned}&\langle V_{j_1,m_1,\bar m_1}V_{j_2,m_2,\bar m_2}V_{j_3,m_3,\bar m_3}\rangle = \\ &= \frac{n}{(2\pi)^3}(\nu)^{j_1+j_2+j_3+1}F(j_1,m_1,\bar m_1;j_2,m_2,\bar m_2;j_3,m_3,\bar m_3)\times \\ &\times\frac{G(-j_1-j_2-j_3-2)G(j_3-j_1-j_2-1)G(j_2-j_1-j_3-1)G(j_1-j_2-j_3-1)}{G(-1)G(-2j_1-1)G(-2j_2-1)G(-2j_3-1)}\,,\end{aligned} \tag{3.2}$$

where the function $G(j)$ defined in [44, 37] satisfies

$$\begin{aligned}G(j) &= G(-j-n-1)\,,\\ G(j-1) &= \frac{\Gamma(1+\frac{j}{k})}{\Gamma(-\frac{j}{k})}G(j)\,,\\ G(j-n) &= n^{-2j-1}\frac{\Gamma(1+j)}{\Gamma(-j)}G(j)\,.\end{aligned} \tag{3.3}$$

Another function $F(j_i,m_i)$ is defined by the following multiple integral [37]

$$\begin{aligned}&F(j_1,m_1,\bar m_1;j_2,m_2,\bar m_2;j_3,m_3,\bar m_3) = \\ &= \int dx_1^2dx_2^2\; x_1^{j_1+m_1}\bar x_1^{j_1+\bar m_1}|1-x_1|^{-2(j_1+j_2-j_3+1)}\times \\ &\times x_2^{j_2+m_2}\bar x_2^{j_2+\bar m_2}|1-x_2|^{-2(j_2+j_3-j_1+1)}|x_1-x_2|^{-2(j_1+j_2-j_3+1)}\,.\end{aligned} \tag{3.4}$$

In general, it is difficult to represent $F(j_i,m_i)$ in terms of simple functions. However, as shown in [58], in the particular case $j_1+m_1=j_1+\bar m_1=0$, we have a following useful expression

$$\begin{aligned}&F(j_1,m_1,\bar m_1;j_2,m_2,\bar m_2;j_3,m_3,\bar m_3) = \\ &= (-1)^{m_3-\bar m_3}\pi^2\frac{\gamma(-j_1-j_2-j_3-1)\gamma(2j_1+1)}{\gamma(1+j_1+j_2-j_3)\gamma(1+j_1+j_3-j_2)}\cdot \\ &\cdot\frac{\Gamma(1+j_2+m_2)\Gamma(1+j_3+m_3)}{\Gamma(-j_2-\bar m_2)\Gamma(-j_3-\bar m_3)}\,,\end{aligned} \tag{3.5}$$

where $\gamma(x)\equiv\Gamma(x)/\Gamma(1-x)$. For example, if we set $j_1=m_1=\bar m_1=0$ in (3.2) and apply the formula (3.5) , then we can check that the three point functions are reduced[23] to the two point function (3.1).

[21] Here we omitted the delta functions $\delta(j-j')\delta^2(m+m')$ when we consider the two point function for $V_{j,m,\bar m}$ and $V_{j',m',\bar m'}$.

[22] Again we suppressed the delta functions $\delta^2(m_1+m_2+m_3)$.

[23] Here we need to replace the divergence $\frac{\Gamma(0)}{2\pi}$ with $\delta(j-j)$.

3.2 Two and three point functions in twisted coset theory

We would like to compute the two and three point functions of tachyon vertex operators $V_j, \tilde{V}_j, T_j$ and $\tilde{T}_j$. The essential parts of the correlation functions are obviously those of the bosonic $\mathrm{SL}(2,\mathbb{R})$ WZW model[24] given by (3.1) and (3.2). In the D^-_j representation, the tachyon vertex operators in the twisted coset model can be expressed in the Ramond state description as follows[25]

$$V_j = \Phi_{j,j}\,, \qquad \tilde{V}_j = \Phi_{j,-j-1}\,, \qquad T_{j+1} = \Phi_{j,j+1}\,, \qquad \tilde{T}_{j+1} = \Phi_{j,-j}\,. \tag{3.6}$$

To make a physical vertex in the closed string we need to combine both the left and right-moving sector. Since the momentum modes in the $c<1$ string side correspond to the winding modes in the coset model, we can impose the restriction $m=\bar{m}$. Thus the correlators of $\Phi_{j,m}$ are equal to those of $V_{j,m,m}$. The operators $V_j, \tilde{V}_j, T_{j+1}$ and $\tilde{T}_{j+1}$ are tachyon vertex operators in the $c<1$ string with the ϕ momentum $p_\phi = i\sqrt{\frac{2}{n}}(j+\frac{1}{2})$ as can be seen[26] from (2.29), (2.33), (2.35) and (2.36). The X momentum of V_j and T_{j+1} is $p_X = \sqrt{\frac{2}{n}}(j+\frac{1}{2})$, while the momentum of $\tilde{V}_j$ and $\tilde{T}_{j+1}$ is $p_X = -\sqrt{\frac{2}{n}}(j+\frac{1}{2})$.

Let us first compute the three point functions in the twisted theory. To do this we can replace two of the three operators with the Ramond ones and the other one with the same one as in the twisted theory (i.e. NS operator) following the general principle [29]. Consider the three point functions $\tilde{C}(j_1,j_2,j_3) \equiv \langle \tilde{V}_{j_1}\tilde{V}_{j_2}\tilde{V}_{j_3}\rangle$ in the twisted theory. They can be computed from the three point functions in the untwisted theory

$$\tilde{C}(j_1,j_2,j_3) = \langle \Phi_{j_1-\frac{n}{2},-j_1+\frac{n}{2}}\ \Phi_{j_2,-j_2-1}\ \Phi_{j_3,-j_3-1}\rangle\,. \tag{3.7}$$

Indeed the J^3_0 charge conservation $j_1+j_2+j_3 = -2+\frac{n}{2}$ agrees with the momentum conservation in the X direction

$$p^1_X + p^2_X + p^3_X - i\frac{Q(X)}{2} = 0\,. \tag{3.8}$$

Then we can find the explicit expressions of the three point functions from the formula (3.2). Also the function $F(j_i,m_i)$ is simplified owing to the formula (3.5). In the end, we find

$$\tilde{C}(j_1,j_2,j_3) = \frac{\Gamma(0)^2}{2\pi\nu}\cdot\gamma\left(1-\frac{2j_1+1}{n}\right)\cdot\gamma\left(1-\frac{2j_2+1}{n}\right)\cdot\gamma\left(1-\frac{2j_3+1}{n}\right). \tag{3.9}$$

This result remarkably agrees with the three point functions in the $c<1$ string up to an overall (divergent) constant factor.[27] To see this, remember that the three point functions in the $c<1$ string is just a constant except the non-local factor (or the leg-factor) for each

[24] In this paper we do not consider the winding number violating amplitudes discussed in e.g. [59]. The author would like to thank Gaston Giribet and Yu Nakayama very much for pointing out this point. In this case, the fermion number (or H momentum) is conserved as we can check in all examples discussed in this subsection.

[25] In the D^+_{-j-1} representation, we can find $V_j = \Phi_{-j-1,j}$, $\tilde{V}_j = \Phi_{-j-1,-j-1}$, $T_{j+1} = \Phi_{-j-1,j+1}$ and $\tilde{T}_{j+1} = \Phi_{-j-1,-j}$.

[26] Notice that here we remembered the relation of the convention in this section and the one in section 2, i.e. $\phi \to -\phi$.

[27] This factor is given by $\frac{\Gamma(0)^2}{2\pi}(\nu)^{(1-n)/n}$. Of course, it can be again included in the non-local factors for the three vertex operators.

(incoming) particles ($i = 1, 2, 3$) [56, 14]

$$\mu^{-\sqrt{\frac{n}{2}}p_X^i} \cdot \gamma\left(1 + \sqrt{\frac{2}{n}}p_X^i\right) = \nu^{j_i+1/2} \cdot \gamma\left(1 - \frac{2j_i+1}{n}\right). \tag{3.10}$$

Here we identified the constant ν with the cosmological constant μ in the Liouville theory. This is because ν is proportional to the coefficient of the screening operator (2.11) [37, 57]. The non-local factor $\gamma(1 + \sqrt{\frac{2}{n}}p_X^i)$ is exactly the same as the leg-factor[28] in the $c < 1$ string.

In the similar way we can analyze other three point functions. For example, consider the three point functions $C(j_1, j_2, j_3) = \langle V_{j_1} V_{j_2} V_{j_3} \rangle$ in the twisted theory. This is essentially reduced to

$$C(j_1, j_2, j_3) = \langle \Phi_{-j_1-\frac{n+2}{2}, j_1+\frac{n+2}{2}} \ \Phi_{j_2,j_2} \ \Phi_{j_3,j_3} \rangle \,, \tag{3.11}$$

where the first one corresponds to the (2.29), and the second and third one to the Ramond operator (2.24). Indeed the J_0^3 charge conservation $j_1 + j_2 + j_3 + \frac{n+2}{2} = 0$ agrees with the momentum conservation (3.8) in the $c < 1$ string. We can again compute the correlators by applying (3.2) and (3.5). It is given by

$$C(j_1, j_2, j_3) = \frac{1}{2\pi n} \cdot \nu^{-2j_1-n-1} \cdot \gamma\left(1 + \frac{2j_1+1}{n}\right). \tag{3.12}$$

Since we can obviously express this by the three non-local factors times a constant, it is consistent with the three particle scattering in the $c < 1$ string. However, one may worry about the asymmetry in (3.12) with respect to the permutation of each particle. This is resolved if we replace the operators $\Phi_{-j_1-\frac{n+2}{2}, j_1+\frac{n+2}{2}}$ with the dual ones $\Phi_{j_1+\frac{n}{2}, j_1+\frac{n+2}{2}}$ using the equivalence (2.34). Then the three point functions simply become

$$C'(j_1, j_2, j_3) \equiv \langle \Phi_{j_1+\frac{n}{2}, j_1+\frac{n+2}{2}} \ \Phi_{j_2,j_2} \ \Phi_{j_3,j_3} \rangle = \frac{n\Gamma(0)}{2\pi} \,, \tag{3.13}$$

where we applied the relation (2.34) to (3.12). Indeed we can see this is symmetric as in the $\tilde{V}_j$ case (3.9). One interesting feature is that this time we do not have any momentum dependent factor like (3.10). This is not surprising because the vertex V_j has the opposite chirality to $\tilde{V}_j$ and they can have different normalizations. This suggests that we should adjust the normalization of V_j and others so that the non-local transformation for the $\tilde{V}_j$ does not depend on the momentum. In this case we have the complete agreement with the $c < 1$ string theory including the leg factors.

Two point functions in the topologically twisted theory are the same as those of the corresponding Ramond operators in the untwisted theory. We can find the nontrivial two point functions (reflection amplitudes) from (3.1)

$$\langle \tilde{V}_j T_{j+1} \rangle = R(j, -j-1, -j-1) \,, \qquad \langle V_j \tilde{T}_{j+1} \rangle = R(j, j, j) \,, \tag{3.14}$$

[28] The leg-factor (3.10) in the $c < 1$ string was first computed in [14] by comparing string theory S-matrix [56] with the matrix model dual (1.1). At $b = 1$ (or $n = 1$), it is obviously reduced to the familiar leg-factor [2] in the $c = 1$ string.

as well as the trivial ones

$$\langle V_j T_{-j}\rangle = \langle \tilde{V}_j \tilde{T}_{-j}\rangle = 1\,. \tag{3.15}$$

Here $R(j,m,m)$ are explicitly given by

$$\begin{aligned} R(j,-j-1,-j-1) &= n\nu^{2j+1}\Gamma(0)\cdot\gamma\left(1-\frac{2j+1}{n}\right),\\ R(j,j,j) &= \frac{\nu^{2j+1}}{n\Gamma(0)}\cdot\gamma\left(-\frac{2j+1}{n}\right). \end{aligned} \tag{3.16}$$

These two point functions again are consistent with those in the $c<1$ string up to non-local factors (or leg-factors). The dependence of $\nu(=\mu)$ agrees with the scaling of μ in the Liouville theory [60]. The first one in (3.16) looks consistent with the (3.10); i.e. the factor $\nu^{j+1/2}\Gamma(0)\cdot\gamma\left(1-\frac{2j+1}{n}\right)$ can be regarded as the non-local factor for the vertex $\tilde{V}_j$ (compare this with (3.9)) . The one[29] for T_{j+1} is $n\nu^{j+1/2}$. We can also have the agreement by assigning the factor $\frac{\nu^{j+1/2}}{n\Gamma(0)}\gamma\left(-\frac{2j+1}{n}\right)$ and $\nu^{j+1/2}$ to $\tilde{T}_{j+1}$ and V_j, respectively.

4. Topological Landau-Ginzburg model

Until now, we have checked the physical spectrum and the three point functions in the twisted $\mathrm{SL}(2,\mathbb{R})_{n+2}/\,\mathrm{U}(1)$ model agree with those in the non-minimal $c<1$ string. To compute more general interactions , i.e. $n(>3)$ point functions in the coset model side, it is one of the easiest way to utilize its topological LG model description. The twisted $\mathrm{SL}(2,\mathbb{R})_{2+n}/\,\mathrm{U}(1)$ model is expected to be equivalent to the topological LG model with the potential

$$W = -\frac{\mu}{n}X^{-n}\,, \tag{4.1}$$

as conjectured in [36]. It can also be regarded as an extension of the well-known relation between the $N=2$ $\mathrm{SU}(2)_k/\,\mathrm{U}(1)$ coset and the $N=2$ LG model with potential $W=X^{k+2}$ to the negative values of k. It will also be useful to notice that the twisted $\mathrm{SU}(2)_k/\,\mathrm{U}(1)$ model or equally the twisted $N=2$ minimal model [48] is known to be equivalent to the minimal $(1,n)$ string [61, 62]. The relation between the twisted coset model and the LG model (4.1) can also be understood from the mirror symmetry or a supersymmetric version of FZZ duality [31, 63]. In this viewpoint the LG model is the B-model mirror description for the A-model topological string on $\mathrm{SL}(2,\mathbb{R})_{n+2}/\,\mathrm{U}(1)$ defined in section 2.2.

Below we will calculate various tree level interactions in the topological LG model (4.1) and check that it agrees with the scattering amplitudes in the $c<1$ string. For $n=1$ case the computations were performed in [39, 40] and they agree with those in the $c=1$ string. Since our analysis here is a natural extension of these results, the discussions in this section will be very brief. In the end, our results in this section again provide a further evidence that supports the equivalence of the twisted coset model and the non-minimal $c<1$ string.

[29] Even though the non-local factors we found for $\tilde{V}_j$ and T_j are asymmetric, we do not think this is problematic just because $\tilde{V}_j$ and T_j have different ghost numbers and their origins are also completely different as is clear from section 2.4.

4.1 Scattering amplitudes

We can find the following correspondence between a closed string tachyon operator T_k and a operator in the twisted LG theory

$$T_k = X^{k-n}\,, \tag{4.2}$$

where k is an arbitrary integer and is proportional to the momentum. It has the U(1) ghost charge $q_k = 1 - k/n$. The momentum conservation is given by

$$\sum_{i=1}^{N} k_i = 2(n-1)\,. \tag{4.3}$$

This comes from the U(1) ghost charge conservation $\sum_i (q_{k_i} - 1) = (g-1)(3-\hat{c})$ in the topological gravity description [61, 34]. Then we get the three point function (setting $\mu = 1$) by applying the standard residue formula in the topological LG model [62]

$$\langle T_{k_1} T_{k_2} T_{k_3} \rangle = \int dX\, \frac{T_{k_1} T_{k_2} T_{k_3}}{W'} = \delta_{k_1+k_2+k_3-2n+2,0}\,. \tag{4.4}$$

In order to compute the four point function, we need to take into account of the contact terms [64, 65]

$$C_W(T_k, T_{k'}) = \frac{d}{dX}\left(T_k T_{k'}/W'\right)_{-} = (k+k'+1-n)\theta(-k-k'+n-1)T_{k+k'}\,, \tag{4.5}$$

where the symbol $_{-}$ means that we take only the negative power part [39, 40]. Thus the four point function is given by the following expression including the contributions from contact terms

$$\begin{aligned}
\langle T_{k_1} T_{k_2} T_{k_3} T_{k_4} \rangle &= \frac{\partial}{\partial t_4} \langle T_{k_1} T_{k_2} T_{k_3} \rangle_{W+t_4 T_{k_4}} + \langle C_W(T_{k_4}, T_{k_1}) T_{k_2} T_{k_3} \rangle + \langle T_{k_1} C_W(T_{k_4}, T_{k_2}) T_{k_3} \rangle + \\
&\quad + \langle T_{k_1} T_{k_2} C_W(T_{k_4}, T_{k_3}) \rangle \\
&= \delta_{k_1+k_2+k_3+k_4-2n+2,0} \times \\
&\quad \times \frac{1}{2}\left[(n+1) - |k_1+k_4+1-n| - |k_2+k_4+1-n| - |k_3+k_4+1-n|\right].
\end{aligned} \tag{4.6}$$

It is easy to check that this expression (4.6) is symmetric under any exchange of the four particles. When we restrict (4.6) to the kinematical region of $1 \to 3$ scattering [56] i.e. $k_{2,3,4} > n-1$ and $k_1 < n-1$, we find the following simplified result

$$\langle T_{k_1} T_{k_2} T_{k_3} T_{k_4} \rangle = (k_1+1)\delta_{k_1+k_2+k_3+k_4-2n+2,0}\,. \tag{4.7}$$

Furthermore, in the same way, we can compute the five particle scattering amplitudes by perturbing potential infinitesimally. For the kinematical region of $1 \to 4$ scattering, we obtain[30]

$$\langle T_{k_1} T_{k_2} T_{k_3} T_{k_4} T_{k_5} \rangle = \delta_{k_1+k_2+k_3+k_4+k_5-2n+2,0} \cdot (k_1+1)(k_1+n+1)\,. \tag{4.8}$$

[30]To see this we note that the five point function can be reduced to four point functions via the derivative of the perturbed ones as in (4.6). This is correct when the perturbation is a primary field and it requires $k > 0$. Indeed, the tachyon fields $T_{k_{2,3,4}}$ satisfy this constraint. Also one more useful fact is that the contact term only appears between the operators $T_{k_{2,3,4}}$.

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In general, we get

$$\langle T_{k_1} T_{k_2} \cdots T_{k_N} \rangle = \left(n \frac{d}{d\mu} \right)^{N-3} \mu^{-(k_1+1)/n}, \tag{4.9}$$

making the μ dependence explicit. Though the momentum dependent factor $\mu^{-(k_1+n-1)/n}$ is missing in the above computation, we can see that it comes from the explicit identification of negative k states T_k with the gravitational descendants[31] as in the $n = 1$ case [39, 40].

Now let us compare this with the string theory results. The relation between p_X in the $c < 1$ string (we assume $\alpha' = 2$ as in section 3.2) [56, 14] and the integer k can be found from the momentum conservation (4.3) as follows

$$p_X = \frac{k}{\sqrt{2n}}. \tag{4.10}$$

This relation can also be understood from the comparison of cosmological constants $|W|^2 \sim \mu e^{\sqrt{2n}\phi}$, which means[32] $|X| \sim e^{-\phi/\sqrt{2n}}$. Then we can identify the compactification radius with $R = \sqrt{2n}$. Assuming $b = \sqrt{n}$, we can check that the previous amplitudes in the LG theory agree with the string theory [56] or matrix model results [14] up to the following energy dependent factor (or leg-factor) for each particle.[33]

$$\frac{\Gamma(1 - \sqrt{\frac{2}{n}} p_X)}{n \cdot \Gamma(\sqrt{\frac{2}{n}} p_X)}. \tag{4.11}$$

In this way we have shown that the topological LG model (4.1) describes the $c < 1$ non-minimal string ($b = \sqrt{n}$) at the radius $R = \sqrt{2n}$.

In this computation, as we have seen, the momentum dependent factor or the leg factor (4.11) in the $c < 1$ string is what we should put by hand. This was also true for the computation of the three point functions for V_j (3.11) obtained from the direct computations in the coset model. These are not problematic since the factor can be removed by a field redefinition for the particle. Nevertheless, it would be intriguing to notice again that the the three point functions for $\tilde{V}_j$ (3.9) include the same non-local factor as the one in the $c < 1$ string. This suggests that we can determine the field redefinition by adjusting the other operators to the canonical one $\tilde{V}_j$.

5. Summary and discussions

5.1 Equivalence for positive integer n

We have shown the following equivalence[34] between the non-minimal $c < 1$ string and the

[31] Here we may identify them with the descendant of the dual cosmological operator so that the power of $\tilde{\mu} = \mu^{1/n}$ is integer.

[32] Even though we used the same symbol X for both the euclidean time (in section 2) and the LG field (in section 3), which are completely different from each other.

[33] One way to find this factor in the LG theory is to fix the overall normalization of the two point functions by rewriting them in terms of disk amplitudes. They are computed in [66] for the SU(2)/ U(1) case. Its extension to our SL(2, $\mathbb{R}$)/ U(1) can be done via the simple continutation of the level $n \to -n$. The author would like to thank Cumrun Vafa very much for explaining this point

[34] Here again we assume $\alpha' = 2$.

twisted $N=2$ $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model or the equivalent topological LG model for integer n

$$\begin{aligned}
&\text{Nonminimal } c = 1-\frac{6(n-1)^2}{n} \quad \text{String at radius } R=\sqrt{2n} \\
&\simeq \text{Twisted A}-\text{model on } \left(\frac{\mathrm{SL}(2,\mathbb{R})_{2+n}}{\mathrm{U}(1)}\right)/\mathbb{Z}_n \quad \left(\text{radius } R_A=\sqrt{\frac{2}{n}}\right) \\
&\simeq \text{Topological LG model } W=-\frac{\mu}{X^n}\,,
\end{aligned} \tag{5.1}$$

where the $\mathbb{Z}_n$ projection is the translation by $\frac{2\pi}{\sqrt{n/2}}$ in the circle direction of the cigar (see e.g. [31]). Notice that the topological (B-model) twisted LG model in (5.1) is expected to be equivalent to that of the $N=2$ Liouville theory via the supersymmetric version of the FZZ duality or equally the mirror symmetry [31]. The main point is the relation between the $c<1$ string and either of the two topological models. Notice that in the above model (5.1) the compactification radius $R=\sqrt{2n}$ is consistent[35] with the string coupling constant $g_s = e^{i(\sqrt{n/2}-1/\sqrt{2n})X}$.

We have checked this claim by computing the physical spectrum and the scattering amplitudes (or equally correlation functions) in the twisted theories and comparing them with those in the $c<1$ string. In particular we directly calculated the three point functions in the twisted $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model in terms of the untwisted theory. This reveals the structure of the momentum dependent factor (or leg factor) for each tachyon vertex. We found that this also essentially agrees with that of the $c<1$ string.

5.2 Equivalence in more general cases

Up to now, in the most of the discussions, we have assumed that n is a positive integer. One important subtle point in the non integer case is that the coupling constant dependence $g_s = e^{i(\sqrt{n/2}-1/\sqrt{2n})X}$ does not seem to respect the periodicity of the compactification radius $R=\sqrt{2n}$. However, many results seem to make sense even if $n>0$ is not an integer. For example, the computation of the physical spectrum in section 2 and the scattering amplitudes in section 3 and 4 can be done for general n and we get the same results. Thus it is natural to think that we can continuously change the value of level n of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset. These observations suggest that we may be able to define the $c<1$ string at the specific radius by the topological twist of $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model in the following way

$$\begin{aligned}
&\text{Nonminimal } c = 1-\frac{6(n-1)^2}{n} \quad \text{String at radius } R=\sqrt{\frac{2}{n}} \\
&\simeq \text{Twisted A}-\text{model on } \frac{\mathrm{SL}(2,\mathbb{R})_{2+n}}{\mathrm{U}(1)} \quad (\text{radius } R_A=\sqrt{2n})\,.
\end{aligned} \tag{5.2}$$

We can also take the $\mathbb{Z}_m$ (m is a positive integer) identification in the circle direction; the special case $n=n'$ will be reduced to (5.1). Also notice that when n is an integer the model defined in (5.2) is also equivalent to the $\mathbb{Z}_n$ quotient of the LG model $(W=-\frac{\mu}{X^n})/\mathbb{Z}_n$. In this case the $\mathbb{Z}_n$ projection restricts the tachyon operators $T_k = X^{k-n}$ to the particular momenta $k\in nZ$.

[35]Refer also to [67] for a similar compactification in the presence of an imaginary linear dilaton gradient. In this case the time-like coordinate is compactified, while in our case the space-like coordinate X is so.

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When n is rational i.e. $n = p/q$ in terms of the coprime integers p, q, the $c < 1$ string has the same central charge as the minimal (p, q) model. Thus we may call the model a non-minimal (p, q) model. For example, if we consider the (critical) 2d black hole in type 0 string (i.e. $n = 1/2$) and perform the topological twist, then the result is equivalent to the non-minimal $(1, 2)$ model.

The exchange $n \leftrightarrow 1/n$ in the twisted $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ theory does not change the matter central charge in its equivalent $c < 1$ string theory while the radius R is replaced with nR. In particular, we can find that the twisted $\mathrm{SL}(2,\mathbb{R})_{2+1/n}/\mathrm{U}(1)$ model is equivalent to the twisted $(\mathrm{SL}(2,\mathbb{R})_{2+n}/\mathrm{U}(1))/\mathbb{Z}_n$ model in (5.1) when n is an integer. This relation comes from the $b \to 1/b$ duality of the Liouville sector in the $c < 1$ string [68]. This should be related to the supersymmetric FZZ-duality [31] that the $N = 2$ Liouville term (when its absolute valued square is taken, its ϕ dependence is $\sim e^{\sqrt{2n}\phi}$) and the $\mathrm{SL}(2,\mathbb{R})$ screening operator ($\sim e^{\sqrt{\frac{2}{n}}\phi}$) are dual to each other and this corresponds to the $b \to 1/b$ duality in the $c < 1$ string.

It will also be intriguing to consider the case $n < 0$. The similar problem in the (untwisted) bosonic $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ CFT was discussed in [69] from the viewpoint of time-dependent background in string theory. In the non-critical string description of our model, the matter sector in the world-sheet theory is described by the time-like Liouville theory [70, 71] because we now have the screening potential in the time-direction after the Wick rotation [14]. Thus the model becomes very close to the minimal model. Indeed when n is a negative integer, the LG potential becomes $W = X^{|n|}$. Thus it is the same as the twisted $N = 2$ minimal model (or the coset $\mathrm{SU}(2)_n/\mathrm{U}(1)$), which is known to be equivalent to the minimal $(1, |n|)$ string or the topological gravity coupled to the $|n|-$th minimal matter [61, 62, 36, 32].

To find analogous twisted SCFT models, such as some similar coset models, which are equivalent to the $\hat{c} = 1$ or $\hat{c} < 1$ type 0 string [7 − 9] will be another interesting problem (refer to e.g. [72] for recent relevant discussions). An important new ingredient in this case will be the existence of the RR-flux backgrounds [8, 73, 19, 20]. We leave this as a future problem.

We would also like to point out a quite recent paper [74], which appeared on the web after we finished the main computations in the present paper. In this paper, an explicit relation has been found between the correlation functions in the bosonic $\mathrm{SL}(2,\mathbb{R})$ WZW model (for arbitrary values of the level) and those in the Liouville theory. This seems to be closely related to our relation between the $N = 2$ twisted coset model and the $c < 1$ string since the relevant Liouville theory looks identical. It may help us to prove our claim for more general correlation functions, though we will leave the detailed study of this issue for a future publication.

5.3 Matrix model dual and Landau-Ginzburg potential

The matrix model dual of the $c < 1$ string is described by the fermi surface (setting $b = \sqrt{n}$ in (1.1)) [14]

$$(W_{(1,0)})^n \cdot W_{(0,1)} = \mu \,, \tag{5.3}$$

where $W_{(1,0)} = (-p - x)e^{-t}/2$ and $W_{(0,1)} = (p - x)e^{-t}/2$ are the elementary conserved charges.

When n is an positive integer, the circle compactification in the topological models corresponds to the periodicity $t \sim t + (1+n)\pi i$ in the time-direction.[36] As can be seen from (2.38) and the asymptotic behavior $x \sim e^{-\phi}$ [2], the compactification does not affect $W_{(1,0)}$ and $W_{(0,1)}$. Thus the $c<1$ string , for an integer n, is expected to be described by the time-dependent background (5.3) of $c=1$ matrix model with the compactified euclidean time coordinate (radius[37] $R_M = \frac{1+n}{2}$).

When n is a rational number $n = p/q$, then the matrix model dual seems to be a more complicated one. If we follow the same logic as before, then it is described by the Fermi surface (5.3) with the $\mathbb{Z}_q$ identification $t \sim t + \frac{p+q}{p}\pi i$ and $(W_{(1,0)}, W_{(0,1)}) \sim (W_{(1,0)},\ e^{2\pi i q/p} W_{(0,1)})$. The explicit calculations of physical quantities in this quotient matrix model will be an interesting future problem.

On the other hand, at a formal level, we may apply the LG/CY correspondence [75, 36] to our previous results of the LG model. Then we naively obtain the '$d = 1 + 2/n$ dimensional surface' ($b = \sqrt{n}$),

$$\mu X^{-n} + \sum_{i=1}^{1+1/n} Y_i Z_i = 0\,, \tag{5.4}$$

in the weighted projective space $WP_{(-2,n,n,\cdots)}$. It is easy to see that the space (5.4) has a vanishing first Chern-class. When $n = 1$, this space (5.4) is the same as the conifold as is well-known [36]. Even though generally d is fractional, we can obviously see that the LG model corresponding to (5.4) is indeed the same as (4.1). If we consider fractional numbers for n as in section 5.2, d can be an integer. If we set $Y_1 = 1$ by gauge fixing and neglect 'winding modes' Y_i, Z_i ($i = 2, 3, \cdots$), then we can find $X^n Z_1 = \mu$, which is the same as (5.3). We can think this is the natural extension of the similar result [36] for $n = 1$ or $c = 1$ string.

Finally, it will be very helpful to uncover any relation between the critical 2d black hole in type 0 string and its twisted version as the ten dimensional type II string on Calabi-Yau space is related to the topological string on the same space. The twisted coset model is equivalent to the non-minimal $(1,2)$ model or equally $c = -2$ string as we have argued in section 5.2. The matrix model dual of the 2d type 0 black holes is already proposed [18, 21, 24] based on the bosonic string counterpart [17]. Thus we can compare it with the matrix model dual for the twisted theory which we have found just before. This will reveal the meaning of twisting in terms of the matrix model. On the other hand, a seemingly related equivalence has recently been found in [76] that the $\hat{c} = 0$ type 0 string [9] is equivalent to the $\mathbb{Z}_2$ orbifold of $c = -2$ (bosonic) string. This issue will also be an important future direction.

Acknowledgments

I would like to thank G. Giribet, Y. Hikida, Y. Nakayama, Y. Sugawara, H. Takayanagi,

[36]The imaginary factor i means that the compactification should be taken in the euclidean time direction as we did in the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ model.

[37]Notice that in the matrix model side we assume $\alpha' = 1$ following the standard notation.

S. Yamaguchi and C. Vafa very much for stimulating discussions and enlightening explanations. I am also grateful to A. Adams, J.L. Karczmarek, J. McGreevy, T. Nakatsu, N. Ohta, T. Sakai, D. Shih, A. Strominger, S. Terashima, A. Tsuchiya and M. Wijnholt for helpful discussions and comments. I would like to thank the Center of Mathematical Sciences in Zhejiang University for its hospitality. This work was supported in part by DOE grant DE-FG02-91ER40654.

A. Spectral flow in $\mathrm{SL}(2,\mathbb{R})$ WZW model

Here we give a brief review of the spectral flow in the SL(2, $\mathbb{R}$) WZW model [45].[38]. This is defined by the following transformation with the winding number w ($w \in \mathbb{Z}$) which preserves the bosonic SL(2, $\mathbb{R}$) (level $k = n + 2$) current algebra

$$J^{3\prime}_n = J^3_n + \frac{n+2}{2} w \delta_{n,0}\,, \qquad J^{\pm\prime}_n = J^{\pm}_{n\mp w}. \tag{A.1}$$

In the notation in section 2, it is equivalent to the shift $|0\rangle \to e^{\sqrt{\frac{n+2}{2}} w X_3}|0\rangle$. Then the Virasoro operators shift

$$L'_n = L_n - w J^3_n - \frac{n+2}{4}\delta_{n,0}\,. \tag{A.2}$$

It is also easy to see explicitly (by plotting the allowed values of (L_0, J^3_0)) the equivalence

$$D^{\pm, w=\mp1}_j \simeq D^{\mp, w=0}_{-j-\frac{n+2}{2}}\,. \tag{A.3}$$

In the Wakimoto representation the spectral flow corresponds to

$$\beta'_n = \beta_{n+w}\,, \qquad \gamma'_n = \gamma_{n-w}\,, \qquad \partial\phi' = \partial\phi - \frac{n+2}{\sqrt{2n}}\frac{w}{z}\,. \tag{A.4}$$

Its action on a state is

$$|0\rangle_\phi \to e^{\sqrt{\frac{n}{2}} w\phi}|0\rangle\,, \qquad |0\rangle_{\beta\gamma} \to |w\rangle_{\beta\gamma}\,, \tag{A.5}$$

where $|w\rangle_{\beta\gamma}$ is defined by

$$\beta_{n-w}|w\rangle_{\beta\gamma} = \gamma_{n+1+w}|w\rangle_{\beta\gamma} = 0\,, \qquad (n = 0, 1, 2, \cdots)\,. \tag{A.6}$$

If we use the bosonized representation

$$\beta = e^{-\tilde{\phi}}\partial\xi\,, \qquad \gamma = e^{\tilde{\phi}}\eta\,, \tag{A.7}$$

the shifted of vacuum is expressed as

$$|w\rangle_{\beta\gamma} = e^{w\tilde{\phi}}|0\rangle\,. \tag{A.8}$$

The conformal dimension of the operator $e^{l\tilde{\phi}}$ is $\Delta = -l(l+1)/2$.

[38]Notice that the convention of this paper is different from [45] by the sign $j_{MO} = -j_{us}$. Also the one for [37] is such that $j_{GK} = -j_{us} - 1$.

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